2-Disjoint Path Covers in Mesh-Torus(New Trends in Theory of Computation and Algorithm)

Makino, Kozo

数理解析研究所講究録
1489 巻 2006 年 181-187

2-Disjoint Path Covers in Mesh-Torus

牧野 格三 (Kozo Makino)
(makino2@is.titech.ac.jp)
東京工業大学大学院 情報理工学研究科 数理・計算科学専攻
Dept. of Mathematical and Computing Sciences,
Tokyo Inst. of Technology, Japan

Abstract

For a graph G, a *k-disjoint path cover problem*, is finding k vertex-disjoint paths joining k distinct source-sink pairs that cover all vertices in the graph. We call a set containing such k disjoint paths *k-DPC*. For a reculanglar grid graph(we call it *mesh*), it is known that there exists a necessary and sufficient condition for the existence of 1-DPC in mesh of size $\geq 4\times 4$ [3]. In this paper, we treat a mesh-torus M, which is obtained from mesh by adding column- and row- wraparound edges, of size even $\times$ even. More precisely, such M is expressed by $M=(V=V_1\cup V_2, E)$, since mesh-torus is bipartite. For any 2 distinct source-sink pairs in M of size $\geq 4\times 4$: $(s,t),(u,v)$, if two of them are in V_1, the others are in V_2, we show the existence of 2-DPC joining s and t, u and v.

1 Introduction

One of issues in various interconnection networks is finding node-disjoint paths. A node-disjoint path can be used as parallel paths for efficient data routing among vertices. An interconnection networks is modeled as a graph, in which vertices and edges corresponded to nodes and links, respectively. In a graph, a node-disjoint path problem in interconnection networks is called vertex-disjoint path problem(disjoint path problem, for short).

There are three categories of disjoint paths, i.e., one-to-one, one-to-many, and many-to-many. One-to-one type has a single source s and a single sink t. The purpose is finding disjoint paths joining s and t. One-to-many type deals with the disjoint paths joining a single source s and k distinct sinks $t_1,t_2,\cdots,t_k$. Many-to-many type deals with the disjoint paths joining k distinct sources $s_1,s_2,\cdots,s_k$ and k distinct sinks $t_1,t_2,\cdots,t_k$. The works in many-to-many type have a relative paucity because of its difficulty and some results can be found in [5, ?]. All of three types of disjoint paths do not care whether it covers all vertices in the graph or not. A *k-disjoint path cover problem(k-DPCP*, for short) in a graph G is finding disjoint paths containing all vertices in G. We denote a set containing such k disjoint paths is *k-DPC.* If necessary, we denote k-DPC$[G,\ (s_1,t_1),\ (s_2,t_2),\cdots,\ (s_k,t_k)]$. J.H.Park et al. [8] proposed a certain graph G which has a k-DPC (more precisely, f(vertices or/and edges)-fault-free k-DPC) for any source and sink sets.

The case of $k=1$, the problem corresponds to hamiltonian path problem in a graph G, that is, the problem is to find a path joining given s and t covers all vertices in G. For 1-DPCP(hamiltonian path problem), there are some definitions in previous works[6, 8]. A graph

G is *hamiltonian*(resp.*hamiltonian-connected*) if there exists a hamiltonian cycle (resp. if each pair of vertices are joined by a hamiltonian path) in G. Note that if a graph G is hamltonian, then $2 \leq \delta(G)$ where $\delta(G)$ is minimum degree of vertices of G. If G is hamiltonian-connected, then $3 \leq \delta(G)$. The bipartite graph $G = (V_1 \cup V_2, E)$ can not be hamilitonian-connected because there is no hamiltonian path from s to t for $s, t \in V_2$ such that $|V_1| \geq |V_2|$. A bipartite graph G with $|V_1| \geq |V_2|$ is *bihamiltonian-conneced* if one of following are satisfied:
(1) if $|V_1| = |V_2|$, there is a hamiltonian path from s to t for all $s \in V_1$ and $t \in V_2$.
(2) if $|V_1| = |V_2| + 1$, there is a hamiltonian path from s to t for all $s, t \in V_1$.
From the definition above, bihamiltonian-connected is defined, only at $|V_1 - V_2| \leq 1$.
For given a graph G, it is difficult to determine whether G is (bi)hamiltonian-connected or not. Of course, almost of classes of graphs is non-hamiltonian-connected. But a rectangular grid graph(we call it *mesh*) $R(m, n)$ of size $\geq 4 \times 4$ is bihamiltonian-connected[3].

In this paper, we discuss about the case of $k = 2$. For reviewing the problem, we define the 2-disjoint path cover problem formally. $V(P_i)$ means a set of vertices included in the path P_i.

2-Disjoint Path Cover Problem(2-DPCP)
Instance: graph $G = (V, E)$ and $s, t, u, v \in V$.
Question: Find 2 paths P_1, P_2 such that
(i) P_1 connects s and t, P_2 connects u and v.
(ii) P_1 and P_2 are disjoint.
(iii) $V(P_1) \cup V(P_2) = V$.

Like a 1-DPCP, we define some new definitions for 2-DPCP. We call the given four vertices (s, t, u, v) *endpoint*. Given G and four endpoints, $[G, (s, t), (u, v)]$ is *solvable* if there is a 2-DPC$[G, (s, t), (u, v)]$. A graph G is *coverable* if $[G, (s, t), (u, v)]$ is solvable for any pairs of vertices: $(s, t), (u, v)$. Note that if a graph G is coverable, then $3 \leq \delta(G)$. A bipartite graph G is *bi-coverable* if one the followings conditions are satisfied:
(A) if $|V_1| = |V_2|$, $[G, (s, t), (u, v)]$ is solvable for any s, t, u, v such that two endpoints of $\{s, t, u, v\}$ are in V_1, the rest are in V_2.
(B) if $|V_1| = |V_2| + 1$, $[G, (s, t), (u, v)]$ is solvable for any s, t, u, v such that three endpoints of $\{s, t, u, v\}$ are in V_1, the rest is in V_2.

In this paper, we consider a given graph G is a mesh-torus $M(m, n)$, which is obtained from $R(m, n)$ by adding column- and row-wraparound edges. Note that $M(m, n)$ is not bipartite if m or n is odd. Therefore, when m and n is even, we can discuss about whether mesh-torus is bi-coverable or not. In addition, $|V_1| = |V_2|$ is true since m and n is even. So if a graph G is mesh-torus, the only case we consider is the condition (A) of bi-coverable. And our result is following:

Theorem 1.1. *If n and m are even ≥ 4, $M(n, m)$ is bi-coverable.*

For showing the Theorem 1.1, we introduce a certain operation (roughly speaking, insert two columns or rows to mesh-torus) for creating any instance of 2-DPCP from some *"prime"* instance. We show that the existence of 2-DPC is retained after this operation. Then, our claim is proved if all "prime" instance is all bi-coverable.

This paper is organized as follows. In section 2, we explain the definitions and essential facts in previous papers. Next section, we provide the operation and its properties. Then, Theorem 1.1 is showed in section 4. In the final section, we conclude our result again. This paper, some proofs are omitted because of page constraint.

2 Preliminaries

Here we prepare necessary notions and notations, and review some important facts on 1-disjoint path cover problem.

A *mesh* $R(m,n)$, more simply R if m and n are fixed, is a graph of $m \times n$ vertices such that (i) each vertex v corresponds to a grid point (x,y), $x \in \{1,2,...,m\}$ and $y \in \{1,2,...,n\}$, and (ii) each edge corresponds to an edge between a pair of adjacent grid points. We consider that a mesh is identical to its corresponding rectangular subgrid on $\mathbf{R}^2$; vertices and edges are regarded as corresponding grid points and edges. For each vertex v in $R(m,n)$, we use v_x and v_y to denote its x- and y-coordinate respectively. A vertex v is called *even* if $(v_x + v_y)$ is even, and odd otherwise. For our explanation, we assume that all vertices are colored either black or white in the following: color every even vertex by white, and odd vertex by black. We denote V_1 is a set of white vertices, V_2 is $V \backslash V_1$.

A *column-torus* $T_c(m,n)$ is $R(m,n)$ with m column-wraparound edges; $((i,n),(i,1))$ $(1 \leq i \leq m)$. A *mesh-torus* $M(n,m)$ is obtained from $T_c(m,n)$ by adding n row-wraparound edges; $((m,i),(1,i))$ $(1 \leq i \leq n)$. Note that $M(m,n)$ is bipartite graph iff m and n is even, $T_c(m,n)$ is bipartite iff n is even. We assume that all vertices in column- and mesh-torus are colored by same way at mesh. In the follwing, four endpoints are satified that two endpoints are in V_1, the others are in V_2.

Fact 1. *If $m,n \geq 4$, or $m=3$ and n is odd, $R(m,n)$ is bihamiltonian-connected[3].*

Fact 2. *$R(2,n)$ is bihamiltonian-connected except that two endpoints are in same non-boundary row[3].*

Lemma 2.1. *Given a $T_c(1,n)(n \geq 4)$ and a source s, there are two sinks t such that there is a hamiltonian path joining s and t.*

Lemma 2.2. *$T_c(2,n)$ is bihamiltonian-connected.*

Remark. Given $T_c(2,n)(n \geq 4)$ and a source s, the number of candidates of sink t such that there exists a hamiltonian path joining s and t and $t_x = i(i=1,2)$ is at least 2.

Lemma 2.3. *Given a $T_c(1,n)(n \geq 4)$ and two sources $s=(1,s_y), u=(1,u_y)(u_y > s_y)$, there are two pairs (t_1,v_1), (t_2,v_2) such that 2-DPC$[T_c(1,n),(s,t_i),(u,v_i)]$ exists$(i=1,2)$.*

3 Unit Insertion and Its Properties

For proving our theorem, we introduce an operation for expanding a mesh-torus of a given 2-DPCP instance. A *unit insertion* is inserting two columns or rows to a mesh-torus M. Then, the following two lemmas are essential for our theorem..

Lemma 3.1. *Let Y be an instance of 2-DPCP obtained from some $X = [M(m,n),(s,t),(u,v)]$ (m,n are even ≥ 4) by any unit insertion to M. If X is solvable, so is Y.*

In our argument, we define some notions for two column insertions; two row insertions are treated in the same way. For any unit insertion, a inserted line —a betweenness of two adjacent columns— is called *chink*, a inserted $2 \times n$ component is called *river*. Now, X has a 2-DPC. For any unit insertion and a 2-DPC, an edge, is a part of 2-DPC, which moves across the chink is called *bridge*. Let b be the number of bridges. And the lemma is proved by simple analysis about b, i.e.,(1) $b = 0$ and (2) $b \neq 0$. For each case, we can construct a 2-DPC of Y from 2-DPC of X, easily. So the proof of this lemma is omitted.

The *empty column* is a column which has no endpoint.

Lemma 3.2. *Let X be any instance of 2-DPCP $[M(m,n),(s,t),(u,v)]$ (m,n are even ≥ 4). The X is obtained from one of the following instances of 2-DPCP by some unit insertion;*
(A) $[M(4,4),(s',t'),(u',v')]$
(B) $[M(6,l),(s',t'),(u',v')]$ *($l = 4,6,8$) and each side of any empty column are non-empty.*
(C) $[M(8,l),(s',t'),(u',v')]$ *($l = 4,6,8$) and each side of any empty column are non-empty.*

This proof is easily by simple case analysis of distances of endpoints. So the proof is omitted. An instance of 2-DPCP X is *prime*, if X is one of (A), (B), (C). The Figure 1 expresses a ambiguous shapes (i.e., almost rows are omitted) of (B) and (C). In this figure, a column with circle(s) means that the column has endpoint(s).

In the next section, we prove the primes' solvality.

4 Prime Instances

In this section, we show the existence of 2-DPC for all prime instances. Let S_1 be a set of all instances is categorized in (B) and (C) described in the Lemma 3.2, and S_2 be a set of all instances of (A). We denote $[M(m,n),(c_1 - c_2 - \cdots - c_m)]$ $(\Sigma_{i\in[1,m]}c_i = 4, c_i \geq 0(\forall 1 \leq i \leq m))$ is a set of instances of 2-DPCP such that column i has a c_i endpoint(s). For example, the (B)-(1) in the Figure 1 is an element of $[M(6,l),(1\text{-}0\text{-}1\text{-}0\text{-}2\text{-}0)]$. In the follwing, we often say $[M(m,n),(c_1 - c_2 - \cdots - c_m)]$ is bi-coverable if for any four endpoints satisfying such condition described as above, it is solvable.

Lemma 4.1. *For any instance of S_1, it becomes to a some element of $[T_c(3,l),(1\text{-}0\text{-}3)]$.*

It is easily to show by using Facts and Lemmas in section 2. So it is omitted.

Lemma 4.2. *A $[T_c(3,l)$, $(1\text{-}0\text{-}3)]$(l is even) is bi-coverable.*

The proof of the Lemma, is simple case analysis, is omitted.
Remark. If $l = 4$, there is only two case of the position of r; $r_y = 2$ or 4. For each assignment of endpoints, it is easy to show that there is a 2-DPC contains the edge ((3,3),(3,4)).

Lemma 4.3. *A $M(4,4)$ is bi-coverable.*

proof. We prove the claim by case analysis. We divide all instance of 2-DPCP by the order of a number of endpoints. That is, our cases are (1)[$M(4,4)$,(4-0-0-0)], (2)[$M(4,4)$,(3-1-0-0)], (3)[$M(4,4)$,(3-0-1-0)], (4)[$M(4,4)$,(2-2-0-0)], (5)[$M(4,4)$,(2-0-2-0)], (6)[$M(4,4)$,(2-1-1-0)], (7)[$M(4,4)$,(2-1-0-1)], and (8)[$M(4,4)$,(1-1-1-1)].

Case 1 There are only two patterns of the combination of pairs. A 2-DPC for these patterns are in the Figure 2.

Case 2 We can deliver the three endpoints in the column 1 to the column 4 by covering the all vertices in the column 1. Then, we obtain the instance of $[T_c(3,4),(1\text{-}0\text{-}3)]$. By Lemma 4.2, the all instance of this case is solvable.

Case 3 We reset a y-coordinate of some white endpoint in the column 1 is 1. By the Lemma 4.2 (**Remark**), the left 3×4 column-torus has 2-DPC, and it contains a $((3,3)(3,4))$. Then, we cut the connection of that edge, and connect $(3,3)$ to $(4,3)$, $(3,4)$ to $(4,4)$. Next, we connect $(3,4)$ to $(4,4)$ by a path crossing column-wraparound edge. In consequence, we obtain a 2-DPC for $[M(4,4),(3\text{-}0\text{-}1\text{-}0)]$.

Case 4 We can deliver one endpoint in the column 2 to the column 1 and the other to the column 3 by covering all vertices in the column 2. Then, we obtain a instance of $[T_c(3,4),(1\text{-}0\text{-}3)]$. By the same argument of the case 2, all instance of this case is solvable.

Case 5 Without loss of generality, we can set the vertices in column 1 are (a) $[(1,1)$ and $(1,3)]$, and (b)$[(1,1)$ and $(1,2)]$. The case of (a), there are two patterns of the combination of pairs. A 2-DPC for these patterns are in the Figure 3.

The case of (b), there are three further cases of location of the rest two endpoints. We express such three cases in the Figure 4(upper side). Furthermore, we consider an assignment of endpoints for all cases in the figure. A 2-DPC for these patterns are in the Figure 4(under side). So, in the case of 4, all instance of this case is solvable.

Case 6 and **7** We can obtain an instance of $[T_c(3,4),(1\text{-}0\text{-}3)]$ from all instance of these cases by same argument in the case 2 or 4 (the delivered endpoints are in column 2 to column 1 in case 6, are in column 1 to column 2 in case 7).

Case 8 The rest cases which we have to consider is only one; i.e., each column and row have just one endpoint. And there are only two patterns of combination of pairs. The Figure 5 expresses a 2-DPC for these two patterns. □

Therefore, we prove our main theorem.
Theorem 1.1. *If n, and m are even ≥ 4, M(n, m) is bi-coverable.*

5 Conclusion and Remarks

In this paper, we show a bi-coverability of even $\times$ even mesh-torus larger than 4×4. However, there is no idea for another size cases; even $\times$ odd, odd $\times$ odd, since such mesh-torus is not bipartite. But, some instance expressed by $X = [M(\text{odd, even}),(s,t),(u,v)]$ has 2-DPC. So next problem is whether $M(\text{odd, even})$ is coverable or not. Furthermore, we are interested in the class or property of graphs which is (bi-)coverability. By the way, in this paper, we treat only 2-DPCP in mesh-torus. A mesh-torus is 4-regular bipartite graph with some restriction. For k-DPCP, what is a minimum number l such that a l-regular bipartite graph with some restrict(as a mesh-torus) is (bi-)coverable (expanding the definition of coverability of 2-DPCP to k-DPCP)? For example, a mesh-torus is not coverable for 3DPCP since $5 \leq \delta$ is necessary condition where δ is minimun degree of the graph.

Acknowledgement

The author would like to thank Professor Osamu Watanabe for supervising this work, Professor Hiro Ito for hosting an important research meeting about combinatorial games and puzzles at Sep. 2005, and Professor Gisaku Nakamura for helpful advices. Thanks also go to Professor Akinori Kawachi and member of Watanabe's group. for significant discussions.

References

[1] F.Luccio and C.Mugnai, "Hamiltonian Paths on a Rectangular Chessboard", *16th Annual Allerton Conference (1978) pp.73-78.*

[2] Y.Perl and Y.Shiloach, "Finding Two Disjoint Paths Between Two Pairs of Vertices in a Graphs", *Journal of the Association for Computing Machinery 25(1) (1978) pp.1-9.*

[3] A.Itai, C.H.Papadimitriou and J.L.Szwarcfiters, "Hamilton Paths in Grid Graphs", *SIAM Journal on Computing 11(4)(1982) pp.676-686.*

[4] H.Everett, "Finding Hamilton Paths in Non-rectangular Grid Graphs", *Congressus Numerantium 53 (1986) pp.185-192.*

[5] S. Madhavapeddy, I.H. Sudborough "A Topological Property of hypercubes : Node Disjoint Paths", *in Proc. of 2th IEEE Symposium on Parallel and Distributed Processing (1990) pp. 532-539.*

[6] H.C.Kim and J.H.Park, "Fault Hamiltonicity of Two-Dimensional Torus Networks", *Workshop on Algorithms and Computation (WAAC 2000) pp. 110-117.*

[7] S.D.Chen, H.Shen, and R.Topor, "An efficient algorithm for constructing Hamiltonian paths in meshes", *Parallel Computing 28 (2002) pp.1293-1305.*

[8] J.H.Park, H.C.Kim, and H.S.Lim, "Many-to-Many Disjoint Path Covers in a Graph with Faulty Elements", *Proc. of the International Symposium on Algorithms and Computation (ISAAC 2004) pp. 742-753.*

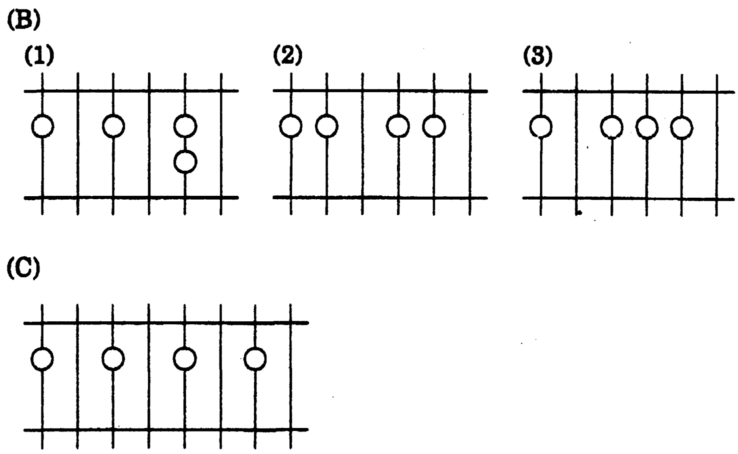

Figure 1: shapes of prime

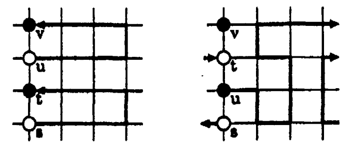

Figure 2: 2-DPCs for the case 1

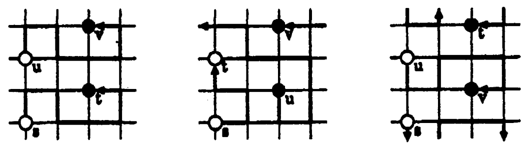

Figure 3: 2-DPCs for the case 5-(a)

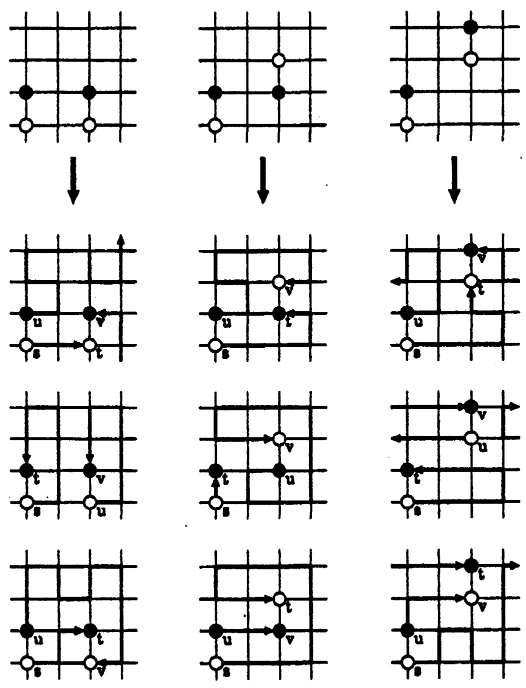

Figure 4: 2-DPCs for the case 5-(b)

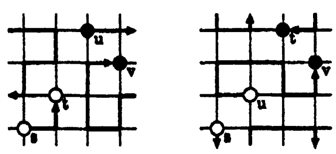

Figure 5: 2-DPCs for the case 8